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LETTER TO THE EDITOR

Noncommutative 3D harmonic oscillator

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Abstract
We find transformation matrices allowing us to express a noncommutative three-dimensional harmonic oscillator in terms of an isotropic commutative oscillator, following the 'philosophy of simplicity' approach. Noncommutative parameters have a physical interpretation in terms of an external magnetic field. Furthermore, we show that for a particular choice of noncommutative parameters there is an equivalent anisotropic representation, whose transformation matrices are far more complicated. We indicate a way to obtain the more complex solutions from the simple ones.

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String theory results [1, 2] have generated a revival of interest in field theory in a noncommutative geometry [3]. A simpler insight into the role of noncommutativity in field theory can be obtained by studying solvable models of noncommutative quantum mechanics [4–6].

Recently, we have presented [7] the description of the noncommutative harmonic oscillator in two dimensions in terms of an isotropic commutative oscillator in an external magnetic field. This interpretation is made possible by the existence of a simple representation of the noncommutative coordinates in terms of the canonical ones. There are many other possible representations of the noncommutative algebra in terms of two Heisenberg algebras [8]. Nevertheless, all of them fall into two groups: those leading to a set of anisotropic oscillators, and others leading to an isotropic oscillator. This correspondence indicates that, in solving an explicit model, one should always look for the simplest form of the solution. As far as two-dimensional models are concerned, the choice of a particular solution may seem to be of less importance. However, it becomes very important in higher dimensions where the set of equations is far more complicated and finding a simple way of solving it becomes essential.

In this letter, we are going to adopt the philosophy of simplicity and point out its advantage in describing the three-dimensional noncommutative harmonic oscillator.

As an introduction we give a brief review of the way in which a noncommutative system can be transformed into an equivalent commutative form. This approach is shown to be equivalent to the introduction of the Moyal $*$-product [9–11], which is the usual way to introduce noncommutativity. One starts with the set of noncommutative coordinates (x, p) of position and momentum satisfying the following commutation relations:

$$[x_k, x_j] = \mathrm{i}\Theta_{kj} \tag{1}$$

$$[p_k, p_j] = \mathrm{i}B_{kj} \tag{2}$$

$$\left[x^k, p_j\right] = \mathrm{i}\delta^k_j \tag{3}$$

where $\mathbf{\Theta}$ and $\mathbf{B}$ are matrices whose elements measure the noncommutativity of coordinate and momenta, respectively. We shall represent noncommutative variables as a linear combination of commutative coordinates $(\vec{\alpha}, \vec{\beta})$ in a six-dimensional phase space

$$\begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{c} \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}. \tag{4}$$

The 6×6 transformation matrix is written in terms of 3×3 blocks $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. The four submatrices satisfy

$$\mathbf{ab}^{\mathrm{T}} - \mathbf{ba}^{\mathrm{T}} = \mathbf{\Theta} \tag{5}$$

$$\mathbf{cd}^{\mathrm{T}} - \mathbf{dc}^{\mathrm{T}} = -\mathbf{B} \tag{6}$$

$$\mathbf{ca}^{\mathrm{T}} - \mathbf{bd}^{\mathrm{T}} = \mathbf{I} \tag{7}$$

following from the commutation relations (1)–(3). $\mathbf{M}^{\mathrm{T}}$ denotes the transpose of a 3×3 matrix $\mathbf{M}$.

As a specific model we choose a three-dimensional, noncommutative, harmonic oscillator described by the Hamiltonian

$$H \equiv \tfrac{1}{2}\left[p_i^2 + x_i^2\right] \tag{8}$$

where we set classical frequency and mass to unity. One can verify that the attempt to solve the system of equations (5)–(7) in full generality (meaning the most general form of the transformation matrices), already in two dimensions, led to a complicated, but still tractable, set of equations [7, 8]. In three dimensions things only get considerably worse. Thus, we apply the above-mentioned philosophy of simplicity. First of all, we note that in three dimensions antisymmetric matrices $\mathbf{\Theta}$ and $\mathbf{b}$ can always be written as

$$\Theta_{ab} \equiv \epsilon_{abc}\theta_c \qquad B_{ab} \equiv \epsilon_{abc}B_c. \tag{9}$$

On physical grounds the isotropic solution, having spherical symmetry, requires the equivalence of all three directions. Therefore, let us impose $\theta_c \equiv \theta$ and $B_c \equiv B$, $\forall\, c$. Furthermore, in analogy with two dimensions, let us choose matrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ to be

$$\mathbf{a} = a\mathbf{I} \qquad \mathbf{c} = c\mathbf{I} \qquad \mathbf{b} = b\mathbf{K} \qquad \mathbf{d} = d\mathbf{K} \tag{10}$$

where $\mathbf{I}$ is the identity and $\mathbf{K}$ is an unknown matrix. The explicit form of $\mathbf{K}$ is found to be

$$\mathbf{K} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{11}$$

Inserting ansatz (10) into (5) and (6) leads to the solutions for the parameters as

$$b = -\frac{\theta}{a} \qquad d = \frac{B}{c}. \tag{12}$$

The remaining equation (7) gives

$$ac + \frac{B\theta}{ac} = 1 \tag{13}$$

which determines the parameters c and d as

$$c = \frac{1}{2a}(1 + \sqrt{1 - 4\theta B}) \equiv \frac{1}{2a}(1 + \sqrt{\kappa}) \tag{14}$$

$$d = \frac{a}{2\theta}(1 - \sqrt{1 - 4\theta B}) \equiv \frac{a}{2\theta}(1 - \sqrt{\kappa}). \tag{15}$$

The three-dimensional solutions follow the same pattern as in two dimensions [7]. The two-dimensional isotropic representation is characterized by the presence of a mixed term in the Hamiltonian, which is reminiscent of the noncommutativity of the system. We find that such a term is also present in this case and is of the form

$$H_{\text{mixed}} = \tfrac{1}{2}(-\theta\alpha_1\beta_2 + B\alpha_2\beta_1) + \cdots. \tag{16}$$

In [7] the mixed term led to the coupling of the noncommutative parameters to the components of the angular momentum operator. Thus, we were able to interpret the noncommutativity as a 'magnetic effect'. In order to reproduce, if possible, the same interpretation in (16) one has to impose the condition

$$B = \theta \tag{17}$$

which allows one to rewrite the mixed term as

$$H_{\text{mixed}} = -\tfrac{1}{2}\theta_i L_i \tag{18}$$

where $\vec{L}$ is the angular momentum operator. We arrive at the isotropic representation of the noncommutative three-dimensional harmonic oscillator:

$$H = h_\alpha(\alpha_i)^2 + h_\beta(\beta_i)^2 - \tfrac{1}{2}\vec{\theta}\cdot\vec{L} \tag{19}$$

$$h_\alpha \equiv \frac{a^2}{2}\left[1 + \frac{1}{4\theta^2}(1 - \sqrt{\kappa})^2\right] \tag{20}$$

$$h_\beta \equiv \frac{\theta^2}{2a^2}\left[1 + \frac{1}{4\theta^2}(1 + \sqrt{\kappa})^2\right] \tag{21}$$

where $\kappa \equiv 1 - 4\theta^2$. Hamiltonian (20) is invariant under spatial rotations. This fact permits us to choose a new set of coordinates with one axis aligned with $\vec{\theta}$. In the rotated frame $\alpha_i \to \mathbf{R}_{ij}\alpha_j$, $\beta_i \to \mathbf{R}_{ij}\beta_j$. Hamiltonian (19) takes a simpler looking form

$$H = h_\alpha(\alpha_i)^2 + h_\beta(\beta_i)^2 - \tfrac{\sqrt{3}}{2}\theta L_\theta. \tag{22}$$

The explicit form of the rotation matrix is given by

$$\mathbf{R} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{pmatrix}. \tag{23}$$

The spectrum of the system is

$$E_{n_+n_-} = \omega\left(n_+ + n_- + n_0 + \tfrac{3}{2}\right) + (n_+ - n_-)\tfrac{\sqrt{3}}{2}\theta \tag{24}$$

where $m \equiv n_+ - n_-$ is the 'magnetic' eigenvalue of the L_θ component of the angular momentum operator, and

$$\omega \equiv 2\sqrt{h_\alpha h_\beta} \tag{25}$$

ω being expressed in terms of units of classical frequency. The explicit solution gives the frequency of the harmonic oscillator equal to the classical frequency. The noncommutative effects are pure magnetic field effects in (19). The results are identical to the two-dimensional case for the special choice $\theta = B$. In three dimensions, however, this choice is imposed by the form of the mixed Hamiltonian and is the only possible solution which gives isotropy of the Hamiltonian. One can rewrite the spectrum as

$$E_{n_+n_-} = \omega_+\left(n_+ + \tfrac{1}{2}\right) + \omega_-\left(n_- + \tfrac{1}{2}\right) + \omega\left(n_0 + \tfrac{1}{2}\right) \tag{26}$$

provided the following identifications are in order:

$$\sqrt{3}\theta = (\omega_+ - \omega_-) \tag{27}$$

$$\omega = \tfrac{1}{2}(\omega_+ + \omega_-). \tag{28}$$

The above spectrum is one of three one-dimensional, anisotropic oscillators. Thus, the three-dimensional noncommutative harmonic oscillator has both isotropic and anisotropic commutative representations. In order to prove the existence of solutions for the transformation matrices of the anisotropic representation, without explicitly solving the complex set of equations (5)–(7), one can proceed in the following way. Let us first find the relation among the commutative coordinates of the two different representations by defining

$$Q_1 = A_1\alpha_1 - A_2\beta_2 \qquad Q_2 = -A_1\alpha_2 + A_2\beta_1 \qquad Q_3 = C\alpha_3 \tag{29}$$

$$P_1 = A_1\alpha_2 + A_2\beta_1 \qquad P_2 = -A_1\alpha_1 - A_2\beta_2 \qquad P_3 = D\beta_3. \tag{30}$$

The parameters in (30) are determined by the requirement that the above redefinitions turn Hamiltonian (19) into its anisotropic form

$$H = \tfrac{1}{2}\omega_+\left(Q_1^2 + P_1^2\right) + \tfrac{1}{2}\omega_-\left(Q_2^2 + P_2^2\right) + \tfrac{1}{2}\omega\left(Q_3^2 + P_3^2\right) \tag{31}$$

which gives the solutions

$$\begin{aligned} A_1 &= \sqrt{\frac{h_\alpha}{\omega}} \qquad & A_2 &= \sqrt{\frac{h_\beta}{\omega}} \\ C &= \sqrt{2}A_1 \qquad & D &= \sqrt{2}A_2. \end{aligned} \tag{32}$$

The relation between the anisotropic coordinates $(\vec{Q}, \vec{P})$, written as a 'column matrix' $\mathbf{Q}$, and isotropic ones $(\vec{\alpha}, \vec{\beta})$ can be written in matrix form as

$$\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} = \begin{pmatrix} A_2\mathbf{L}_1 & A_2\mathbf{L}_2 \\ -A_1\mathbf{L}_2 & A_1\mathbf{L}_1 \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}. \tag{33}$$

The above equation is written in the block form with 3×3 matrices $\mathbf{L}_1, \mathbf{L}_2$ given by

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \qquad \mathbf{L}_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

The relation between the noncommutative coordinates (after rotation) and the isotropic set of solutions can be written in block form as

$$\begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} a\mathbf{R}^{\mathrm{T}} & b\mathbf{K}\mathbf{R}^{\mathrm{T}} \\ d\mathbf{K}\mathbf{R}^{\mathrm{T}} & c\mathbf{R}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}. \tag{35}$$

On the other hand, the anisotropic transformation matrices relate the noncommutative coordinates to $(\vec{Q}, \vec{P})$ as

$$\begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{a}} & \widetilde{\mathbf{b}} \\ \widetilde{\mathbf{d}} & \widetilde{\mathbf{c}} \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}. \tag{36}$$

Comparing (36) to (35) with the help of (33), one obtains solutions

$$\widetilde{\mathbf{a}} = aA_2\widetilde{\mathbf{L}}_1 - bA_1\widetilde{\mathbf{L}}_4 \qquad \widetilde{\mathbf{b}} = aA_2\widetilde{\mathbf{L}}_2 + bA_1\widetilde{\mathbf{L}}_3 \tag{37}$$

$$\widetilde{\mathbf{c}} = dA_2\widetilde{\mathbf{L}}_4 + cA_1\widetilde{\mathbf{L}}_1 \qquad \widetilde{\mathbf{d}} = dA_2\widetilde{\mathbf{L}}_3 - cA_1\widetilde{\mathbf{L}}_2 \tag{38}$$

where the 3×3 matrices $\widetilde{\mathbf{L}}_i$, $i = 1, 2, 3, 4$, are found to be

$$\widetilde{\mathbf{L}}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & 0 & 2\sqrt{2} \end{pmatrix} \qquad \widetilde{\mathbf{L}}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -\sqrt{3} & -\sqrt{2} & 0 \\ \sqrt{3} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \tag{39}$$

$$\widetilde{\mathbf{L}}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & 0 & 2\sqrt{2} \\ \sqrt{2} & \sqrt{3} & -\sqrt{2} \end{pmatrix} \qquad \widetilde{\mathbf{L}}_4 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 \\ -\sqrt{3} & -\sqrt{2} & 0 \end{pmatrix}. \tag{40}$$

Exploiting the explicit solutions (20), (21) and (32), one can rewrite the anisotropic set of solutions in terms of isotropic ones:

$$\begin{aligned} \widetilde{\mathbf{a}} &= \sqrt{\frac{ac}{2}}\left(\widetilde{\mathbf{L}}_1 + \frac{\theta}{ac}\widetilde{\mathbf{L}}_4\right) \qquad & \widetilde{\mathbf{b}} &= \sqrt{\frac{ac}{2}}\left(\widetilde{\mathbf{L}}_2 - \frac{\theta}{ac}\widetilde{\mathbf{L}}_3\right) \\ \widetilde{\mathbf{a}} &= \widetilde{\mathbf{c}} & \widetilde{\mathbf{d}} &= -\widetilde{\mathbf{b}}. \end{aligned} \tag{41}$$

One can verify that the above set of solutions satisfies basic requirements (5)–(7). As already advocated, a comparison of the isotropic solutions with the anisotropic ones confirms the validity of the 'philosophy of simplicity' approach.

One may wonder if there are other solutions leading to the same isotropic representation. Let us assume that the transformation matrices are of the same form as before, but with different matrix elements. For example, the matrices **a** and **c** are $\mathbf{a}_{ij} = a^{(i)}\delta_{ij}$ (no summation over the i index), while matrices **b** and **d** are given by

$$\mathbf{b} = \begin{pmatrix} 0 & b_{12} & 0 \\ 0 & 0 & b_{23} \\ b_{31} & 0 & 0 \end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix} 0 & d_{12} & 0 \\ 0 & 0 & d_{23} \\ d_{31} & 0 & 0 \end{pmatrix}. \tag{42}$$

Without going into details, the Hamiltonian following from the above solution is the generalization of (19) with different coefficients h_i, $i = 1, \ldots, 6$, multiplying canonical coordinates. The isotropy of the system requires the equivalence of those coefficients for the coordinates α and β, respectively. This requirement inevitably leads to the condition (17). Thus, we conclude that there are no other isotropic solutions different from those described in this letter. We have thus shown that the three-dimensional noncommmutative harmonic oscillator can be represented as an isotropic oscillator coupled to an external magnetic field, generated by space noncommutativity. This representation is based on a very simple set of transformation matrices relating noncommutative to canonical coordinates. An alternative representation is also possible in terms of three one-dimensional anisotropic harmonic oscillators. The second set of solutions is far more complicated and difficult to obtain solving (5)–(7). Nevertheless, we have described an indirect way of finding these solutions. Their explicit form was needed to support the philosophy of simplicity approach described in this letter.

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